KIT Department of Informatics
Institute for Anthropomatics and Robotics (IAR)
High Performance Humanoid Technologies (H^2T)

Robotics I, WS 2024/2025

Solution Sheet 3

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Solution 1 (Differential Inverse Kinematics)

1. Calculating the Jacobian

   The function for the position of the end-effector is given by

$$\mathbf{f}(\mathbf{q}) = \begin{pmatrix} -500\sin(\theta_2)\cos(\theta_3) - 500\cos(\theta_2)\sin(\theta_3) - 500\sin(\theta_2) \\ 500\cos(\theta_2)\cos(\theta_3) - 500\sin(\theta_2)\sin(\theta_3) + 100 + 500\cos(\theta_2) \\ d_1 \end{pmatrix}$$

   The Jacobian is given by

$$J(\mathbf{q}) = \frac{\partial \mathbf{f}}{\partial \mathbf{q}} = \begin{pmatrix} \frac{\partial \mathbf{f}}{\partial d_1} & \frac{\partial \mathbf{f}}{\partial \theta_2} & \frac{\partial \mathbf{f}}{\partial \theta_3} \end{pmatrix} = \begin{pmatrix} 0 & -500\cos(\theta_2+\theta_3) - 500\cos(\theta_2) & -500\cos(\theta_2+\theta_3) \\ 0 & -500\sin(\theta_2+\theta_3) - 500\sin(\theta_2) & -500\sin(\theta_2+\theta_3) \\ 1 & 0 & 0 \end{pmatrix}$$

   Inverting the Jacobian results in

$$J^{-1}(\mathbf{q}) = \begin{pmatrix} 0 & 0 & 1 \\ -\frac{1}{500}\frac{\sin(\theta_2+\theta_3)}{\sin(\theta_3)} & \frac{1}{500}\frac{\cos(\theta_2+\theta_3)}{\sin(\theta_3)} & 0 \\ \frac{1}{500}\frac{\sin(\theta_2+\theta_3)+\sin(\theta_2)}{\sin(\theta_3)} & \frac{1}{500}\frac{-\cos(\theta_2+\theta_3)-\cos(\theta_2)}{\sin(\theta_3)} & 0 \end{pmatrix}$$

2. Joint angular velocities

$$\dot{\mathbf{q}} = J^{-1}(\mathbf{q})\,\dot{\mathbf{x}}$$

$$\dot{\mathbf{q}} = J^{-1}\left(\left(1, 0, \frac{\pi}{2}\right)^\top\right) \cdot (1000, 0, 0)^\top = \begin{pmatrix} 0 & 0 & 1 \\ -\frac{1}{500} & 0 & 0 \\ \frac{1}{500} & -\frac{1}{500} & 0 \end{pmatrix} \cdot \begin{pmatrix} 1000 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix}$$

3. Singularities

   For quadratic matrices, singularities can be determined by setting the determinate to zero. In this case the matrix looses its full rank.

$$\det J(\mathbf{q}) = 500^2 \cdot \sin\theta_3 = 0$$

   The Jacobian is singular for $\theta_3 = 0°, 180°$. For this joint positions the inverse matrix does not exist any more.

Solution 2 (Lagrangian dynamic modeling)

For our robot we assume the following simplifications:

- m_1 and m_2 are definded as point masses in the middle of the respective segments,
- the system is frictionless,
- the robot is installed at a height of $h = 0$.

The equation of motion can be determined with the Lagrangian method according to the following equation:

$$\boldsymbol{\tau} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}}$$

with the Lagrangian function:

$$L = E_{kin} - E_{pot}.$$

1. Kinetic energy for s_1 and s_2

   For s_1, the following kinetic energy results form the translational velocity $\dot{d}$

   $$E_{kin,1} = \frac{1}{2} m_1 \dot{d}^2.$$

   For s_2, the following kinetic energy results from the velocities $(\dot{x}_2, \dot{y}_2)$ and $\dot{\theta}$:

   $$E_{kin,2} = \frac{1}{2} m_2 v_2^2 + \frac{1}{2} J \omega^2 = \frac{1}{2} m_2 \left(\dot{x}_2^2 + \dot{y}_2^2\right) + \frac{1}{2} J \dot{\theta}^2. \quad (1)$$

   The velocities $\dot{x}_2$ and $\dot{y}_2$ result from the temporal derivative of x_2 and y_2:

   $$\begin{aligned} \dot{x}_2 &= \dot{d} + \frac{1}{2} l_2 \dot{\theta} \sin(\theta), \\ \dot{y}_2 &= -\frac{1}{2} l_2 \dot{\theta} \cos(\theta). \end{aligned}$$

   For a rod with negligible radius, the moment of inertia with respect to the center of mass is as follows:

   $$J = \frac{1}{12} m_2 l_2^2$$

   Placed in Equation 1, this results in the kinetic energy for segment s_2:

   $$E_{kin,2} = \frac{1}{2} m_2 \dot{d}^2 + \frac{1}{6} m_2 l_2^2 \dot{\theta}^2 + \frac{1}{2} m_2 l_2 \dot{d} \dot{\theta} \sin(\theta).$$

2. Potential energy for s_1 and s_2

   For the general case, where the origin of the robot is at a height of $h > 0$, the potential energy with the gravitational vector g for s_1 can be described as follows:

$$E_{pot,1} = m_1 g h.$$

   Respectively for s_2

$$E_{pot,2} = m_2 g \left(h - \frac{1}{2} l_2 \sin(\theta) \right).$$

   From $h = 0$ follows:

$$\begin{aligned} E_{pot,1} &= 0 \\ E_{pot,2} &= -\frac{1}{2} m_2 g l_2 \sin(\theta). \end{aligned}$$

3. Lagrange-function and their derivative

   The Lagrangian function is set up from the kinetic and potential energy for s_1 and s_2 according to the following rule:

$$L = E_{kin,1} + E_{kin,2} - E_{pot,1} - E_{pot,2}.$$

   L is given using the previous results:

$$L = \frac{1}{2}(m_1 + m_2)\dot{d}^2 + \frac{1}{6} m_2 l_2^2 \dot{\theta}^2 + \frac{1}{2} m_2 l_2 \dot{d} \dot{\theta} \sin(\theta) + \frac{1}{2} m_2 g l_2 \sin(\theta).$$

The equations of motion consist of the two components τ_1 and τ_2, for the determination of which the following derivatives need to be computed:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{d}}, \quad \frac{\partial L}{\partial d}, \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}}, \quad \frac{\partial L}{\partial \theta}.$$

The derivatives of L for the translational joint with the control signal d are defined as follows:

$$\begin{aligned} \frac{\partial L}{\partial \dot{d}} &= (m_1 + m_2)\dot{d} + \frac{1}{2} m_2 l_2 \dot{\theta} \sin(\theta) \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{d}} &= (m_1 + m_2)\ddot{d} + \frac{1}{2} m_2 l_2 (\ddot{\theta} \sin(\theta) + \dot{\theta}^2 \cos(\theta)) \\ \frac{\partial L}{\partial d} &= 0. \end{aligned}$$

The derivatives of L for the rotational joint with the control signal θ are defined as follows:

$$\begin{aligned}
\frac{\partial L}{\partial \dot\theta} &= \frac{1}{3} m_2 l_2^2 \dot\theta + \frac{1}{2} m_2 l_2 \dot d \sin(\theta) \\
\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} &= \frac{1}{3} m_2 l_2^2 \ddot\theta + \frac{1}{2} m_2 l_2 (\ddot d \sin(\theta) + \dot d \dot\theta \cos(\theta)) \\
\frac{\partial L}{\partial \theta} &= \frac{1}{2} m_2 l_2 \dot d \dot\theta \cos(\theta) + \frac{1}{2} m_2 l_2 g \cos(\theta).
\end{aligned}$$

Therefrom result τ_1 and τ_2:

$$\begin{aligned}
\tau_1 &= \frac{d}{dt}\frac{\partial L}{\partial \dot d} - \frac{\partial L}{\partial d} \\
&= (m_1 + m_2)\ddot d + \frac{1}{2} m_2 l_2 \sin(\theta)\ddot\theta + \frac{1}{2} m_2 l_2 \cos(\theta)\dot\theta^2 - 0 \\
\tau_2 &= \frac{d}{dt}\frac{\partial L}{\partial \dot\theta} - \frac{\partial L}{\partial \theta} \\
&= \frac{1}{3} m_2 l_2^2 \ddot\theta + \frac{1}{2} m_2 l_2 \ddot d \sin(\theta) + \frac{1}{2} m_2 l_2 \dot d \dot\theta \cos(\theta) - \left(\frac{1}{2} m_2 l_2 \dot d \dot\theta \cos(\theta) + \frac{1}{2} m_2 l_2 g \cos(\theta)\right) \\
&= \frac{1}{3} m_2 l_2^2 \ddot\theta + \frac{1}{2} m_2 l_2 \sin(\theta)\ddot d - \frac{1}{2} m_2 l_2 g \cos(\theta).
\end{aligned}$$

The general form of the equation of motion, neglecting friction, is as follows:

$$\boldsymbol{\tau} = M(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{c}(\dot{\mathbf{q}}, \mathbf{q}) + \mathbf{g}(\mathbf{q})$$

The mass-inertia matrix M is obtained by collecting all terms that depend on $\ddot{\mathbf{q}}$. The vector $\mathbf{c}$ (centripetal and Coriolis components) collects all terms that depend on both $\dot{\mathbf{q}}$ and $\mathbf{q}$. The vector $\mathbf{g}$ collects all terms containing gravitational components.

Thus, the equation of motion can be described as follows:

$$\boldsymbol{\tau} = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} = \begin{pmatrix} m_1 + m_2 & \frac{1}{2} m_2 l_2 \sin(\theta) \\ \frac{1}{2} m_2 l_2 \sin(\theta) & \frac{1}{3} m_2 l_2^2 \end{pmatrix} \begin{pmatrix} \ddot d \\ \ddot\theta \end{pmatrix} + \begin{pmatrix} \frac{1}{2} m_2 l_2 \dot\theta^2 \cdot \cos(\theta) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{1}{2} m_2 l_2 g \cdot \cos(\theta) \end{pmatrix}.$$

Note: The generalized forces τ_1 and τ_2 have different units. τ_1 is a force (unit: Newton), while τ_2 is a torque (unit: Newton-meter).

Solution 3 (Python Introduction)

You can find a solution to solve the exercises in the jupyter notebook provided in the git repository.